On *-Representations of a Certain Class of Algebras Related to Extended Dynkin Graphs

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On ∗-representations of a certain class of algebras related to extended Dynkin graphs

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Abstract

To a family of self-adjoint operators with given spectra whose sum is a scalar operator $\lambda I$ we associate a graph $\Gamma$ and a function on $\Gamma$ (character). In this paper, we study such families in the case where the corresponding graph is an extended Dynkin graph. For the special character on an extended Dynkin graph we describe the set of parameters for which such families exist, and give formulas for the irreducible families. We prove that all the irreducible families are finite-dimensional regardless of the choice of the character on $\Gamma$ and the value of $\lambda$.

Key words: family of self-adjoint operators, representations of involutive algebras, extended Dynkin graph, Coxeter functors, rigidity.

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Introduction

1. Let $H$ be a separable Hilbert space, and let $A_1, \ldots, A_n$ be a family of self-adjoint operators in $H$ with fixed finite spectra, such that $A_1 + \cdots + A_n = \lambda I$ for some $\lambda \in \mathbb{R}$. Such a family of operators can be treated as a $*$-representation of a certain algebra with a finite number of generators and polynomial relations. The corresponding algebras and their $*$-representations were studied in a number of recent papers. The interest to such algebras and their $*$-representations is due to their relations with such topics as Horn’s problem and its modifications (see [Ful00] and the references therein), deformed preprojective algebras (see, e.g., [CBH98]), locally scalar representations of graphs (see [KR05]), singular integral operators (see [Vas98]).

2. Considering a family of self-adjoint operators with a pre-defined spectra, for which $A_1 + \cdots + A_n = \lambda I$, we can assume that $\lambda > 0$, and

$$\sigma(A_l) \subset M_l = \{0 = \alpha_0^{(l)} < \alpha_1^{(l)} < \cdots < \alpha_{k_l}^{(l)}\}, \quad l = 1, \ldots, n.$$  

Following [MSV05, MSZ04] we introduce an algebra as follows. Consider a simply-laced non-oriented graph $\Gamma$ consisting of $n$ branches, such that the $l$-th branch has $k_l + 1$ vertices, $l = 1, \ldots, n$, and all branches are connected at a single root vertex. Marking the vertices of the $l$-th branch (excluding the root vertex) by positive numbers $(\alpha_j^{(l)})_{j=1}^{k_l}$ increasing to the root, we get a function

$$\chi = (\alpha_1^{(1)}, \ldots, \alpha_{k_1}^{(1)}; \cdots; \alpha_1^{(n)}, \ldots, \alpha_{k_n}^{(n)})$$

on the graph $\Gamma$ defined in all vertices except for the root (below this function will be called a character on $\Gamma$). The root vertex will be marked by the number $\lambda$ (notice that the term character in other papers is used to denote a function $(\chi, \lambda)$ on the whole graph, including the root vertex. For our needs the given notation is more convenient).

Given a graph $\Gamma$, a character $\chi$ on $\Gamma$ and a positive number $\lambda$, one can construct the following $*$-algebra

$$A_{\Gamma, \chi, \lambda} = \mathbb{C}\langle a_l = a_l^*, l = 1, \ldots, n \mid p_l(a_l) = 0, l = 1, \ldots, n; \sum_{l=1}^{n} a_l = \lambda e \rangle,$$

where $p_l(x) = x(x-\alpha_1^{(l)}) \cdots (x-\alpha_{k_l}^{(l)})$, $k = 1, \ldots, n$. Then the family $A_1, \ldots, A_n$ is a $*$-representation of $A_{\Gamma, \chi, \lambda}$.

3. The properties of the algebra $A_{\Gamma, \chi, \lambda}$ and the structure of its $*$-representations crucially depend on the type of the graph $\Gamma$: they are quite different for the cases where $\Gamma$ is a Dynkin graph, an extended Dynkin graph or none of them.

If $\Gamma$ is a Dynkin graph then the corresponding algebra is finite-dimensional regardless of the choice of $\chi$ and $\lambda$. If $\Gamma$ is an extended Dynkin graph, then the algebra is infinite-dimensional of polynomial growth, and for other graphs (containing an extended Dynkin graph as a proper subgraph) the corresponding algebra is infinite dimensional and has exponential growth [MSV05].
For $^\ast$-representations of $\mathcal{A}_{\Gamma,\chi,\lambda}$, the following natural problems arise: (i) describe the set $\Sigma_{\Gamma,\chi}$ of those $\lambda \in \mathbb{R}$, for which there exist $^\ast$-representations or (i') describe the set $\Sigma_{\Gamma}$ of those $(\chi, \lambda)$, for which there exist $^\ast$-representations; (ii) for $\lambda \in \Sigma_{\Gamma,\chi}$ or $\lambda \in \Sigma_{\Gamma}$, study the structure and properties of irreducible $^\ast$-representations. These problems were studied for algebras associated to Dynkin graphs [KPS05], and for graphs with $k_1 = \cdots = k_n = 1$ and characters $\chi = (1; \ldots; 1)$ on such graphs [OS99, KRS02].

The main subject of this paper is to study the problems (i) and (ii) for extended Dynkin graphs. In this case, there exists a special character $\chi_{\Gamma}$ on the corresponding graph $\Gamma$. For such special characters, the sets $\Sigma_{\Gamma,\chi}$ of those $\lambda$ for which there exist representations of the corresponding algebra and the corresponding $^\ast$-representations were studied in [OS99, MRS04, MSZ04], e.g..

In Section 2 we study $^\ast$-representations of algebras $\mathcal{A}_{\Gamma,\chi_{\Gamma}}$ in the cases where $\Gamma$ is an extended Dynkin graph and $\chi_{\Gamma}$ is the corresponding special character. We present proofs of results on the structure of $\Sigma_{\Gamma,\chi_{\Gamma}}$ announced in [MSZ04] and give some explicit formulas for irreducible $^\ast$-representations.

In Section 3 we consider general characters on extended Dynkin graphs. In particular, we prove that all irreducible $^\ast$-representations of $\mathcal{A}_{\Gamma,\chi,\lambda}$ are finite-dimensional for any $\chi$ and $\lambda$ provided that $\Gamma$ is an extended Dynkin graph (Theorem 9). As a corollary we get that for all $\lambda$ except one point, the operators of irreducible $^\ast$-representation form a rigid family.

1 Preliminaries

1.1 Coxeter functors

We recall the construction of the Coxeter functors (see [Kru02, KRS02, KPS05]).

1. Consider a non-negative operator $A = 0 \cdot P_0 + \alpha_1 P_1 + \cdots + \alpha_m P_m$ with finite spectrum $\sigma(A) \subset \{0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m\}$ (some of the spectral projections may be zero). Introduce the mapping $A \mapsto \hat{A} = \alpha_m I - A$. Then $\hat{A}$ is a non-negative operator with spectrum $\sigma(\hat{A}) \subset \{0 = \hat{\alpha}_0 < \hat{\alpha}_1 < \cdots < \hat{\alpha}_m\}$, where $\hat{\alpha}_0 = \alpha_m - \alpha_m = 0$, $\hat{\alpha}_1 = \alpha_m - \alpha_{m-1}$, $\ldots$, $\hat{\alpha}_{m-1} = \alpha_{m-1} - \alpha_1$, $\hat{\alpha}_m = \alpha_m$.

Let $A_i = \sum_{k=0}^{m_i} \alpha_k^{(i)} P_k^{(i)}$, $0 = \alpha_0^{(i)} < \cdots < \alpha_{m_i}^{(i)}$, $i = 1, \ldots, d$; $\sum_{i=1}^{d} A_i = \lambda I$, $\lambda \geq 0$ (for the Dynkin graphs we have $d = 3$ or $d = 4$).

Obviously, for the operators $\hat{A}_i$ we have

$$\sum_{i=1}^{d} \hat{A}_i = \left( \sum_{i=1}^{d} \alpha_{m_i}^{(i)} - \lambda \right) I.$$

Therefore, we have a mapping from the set of representations of the algebra $\mathcal{A}_{\Gamma,\chi,\lambda}$ into the set of representations of the algebra $\mathcal{A}_{\Gamma,\tilde{\chi},\tilde{\lambda}}$, where $\chi$ is defined by the set $\alpha_k^{(i)}$, $\tilde{\chi}$ is defined by the set $\tilde{\alpha}_k^{(i)}$, $k = 1, \ldots, m_i$, $i = 1, \ldots, d$, and $\tilde{\lambda} = \sum_{i=1}^{d} \alpha_{m_i}^{(i)} - \lambda$. This mapping will be
denoted by $S$ and called the linear reflection. Applying the linear reflection twice, we get the original operators.

2. Let $P_1, \ldots, P_n$ be projections in a Hilbert space $H$ such that $\sum \alpha_k P_k = \lambda I$ for some positive numbers $\alpha_1, \ldots, \alpha_n, \lambda$. We construct another Hilbert space $\hat{H}$ and a family of projections $\hat{P}_1, \ldots, \hat{P}_n, \hat{P}$ in $\hat{H}$ such that $\sum_{k=1}^n \hat{P}_k = I$ and $\hat{P}_k \hat{P} \hat{P}_k = \frac{\alpha_k}{\lambda} \hat{P}_k$ (dilation) as follows.

Let $\sum \alpha_k P_k = \lambda I, \lambda > 0$. Put $H_k = \text{Im} P_k$, and let $O_k: H_k \rightarrow \hat{H}$ be embeddings, so that $O_k \hat{O}_k^* k = P_k$, and $O_k^* O_k = I_{H_k}$; define $\hat{H} = \bigoplus_{k=1}^n H_k$. Consider the operator

$$O = \frac{1}{\sqrt{\lambda}} \left( \begin{array}{c} \sqrt{\alpha_1} O_1^* \\ \vdots \\ \sqrt{\alpha_n} O_n^* \end{array} \right): H \rightarrow \hat{H}$$

then

$$O^* : \hat{H} \rightarrow H, \quad O^* = \frac{1}{\sqrt{\lambda}} \left( \sqrt{\alpha_1} O_1 \ldots \sqrt{\alpha_n} O_n \right)$$

and $O^* O = \lambda^{-1} \sum \alpha_k O_k O_k^* = \lambda^{-1} \sum \alpha_k P_k = I$, i.e., $O : H \rightarrow \hat{H}$ is an isometric embedding.

Put $\hat{P} = OO^*$, then $\text{Im} \hat{P} = H$, and let $\hat{P}_j$ be projections onto orthogonal subspaces $H_j$ in $\hat{H}$, then we have $\sum \hat{P}_j = I$ and $\hat{P}_j \hat{P}_j^* = \frac{\alpha}{\lambda} \hat{P}_j$. Notice that the latter implies $\hat{P}_k = 0$ and therefore, $P_k = 0$ for those $k$, for which $\alpha_k > \lambda$.

3. The dilation procedure described above can be inverted. Indeed, let $\hat{P}_1, \ldots, \hat{P}_n$, $\hat{P}$ be projections in some Hilbert space $\hat{H}$ such that $\sum_{k=1}^n \hat{P}_k = I$, and for some $\alpha_k > 0$, $\lambda > 0$ we have $\hat{P}_k \hat{P}_k^* = \frac{\alpha_k}{\lambda} P_k$ for all $k = 1, \ldots, n$. We describe the contraction procedure.

Define $H = \text{Im} \hat{P}$, and let $O : H \rightarrow \hat{H}$ be the corresponding embedding. Then $O^* O = I_{H}, O O^* = \hat{P}$. Define $P_k = \alpha_k O^* \hat{P}_k O, k = 1, \ldots, n$. Then we have

$$P_k^2 = \frac{\lambda^2}{\alpha_k^2} O^* \hat{P}_k O O^* \hat{P}_k O = \frac{\lambda^2}{\alpha_k^2} O^* \hat{P}_k \hat{P}_k O = \lambda \frac{\lambda}{\alpha_k} \hat{P}_k O = \lambda I.$$

Applying the dilation and contraction procedures consequently, we get a family of projections $P_1, \ldots, P_n$ which is unitary equivalent to the original one.

4. Let the projections $\hat{P}_1, \ldots, \hat{P}_n$, $\hat{P}$ in $\hat{H}$ satisfy $\sum_{k=1}^n \hat{P}_k = I, \hat{P}_k \hat{P} \hat{P}_k = \frac{\alpha_k}{\lambda} \hat{P}_k$. Taking instead of $\hat{P}$ the projection $\hat{P}' = I - \hat{P}$, we get $\hat{P}_k \hat{P}' \hat{P}_k = \frac{\lambda - \alpha_k}{\lambda} \hat{P}_k$ for all $k = 1, \ldots, n$.

Therefore, taking a family of projections $P_1, \ldots, P_n$ with $\sum_{k=1}^n \alpha_k P_k = \lambda I$, then dilating to the family $\hat{P}_1, \ldots, \hat{P}_n, \hat{P}$, then taking $\hat{P}' = I - \hat{P}$, and finally contracting back, we get a family of projections $P'_1, \ldots, P'_n$ in some Hilbert space $H'$, for which

$$\sum_{k=1}^n (\lambda - \alpha_k) P'_k = \lambda I.$$
5. It follows directly from the construction, that \( P_j \perp P_k \) if and only if for the dilated projections we have \( \hat{P}_j \hat{P}_k = 0 \). Therefore, \( P'_j \perp P'_k \) for some \( j, k \) if and only if \( P_k \perp P_j \).

6. Summarizing the constructions above, we get another mapping from the set of representations of \( A_{\Gamma,\chi} \) to the set of representations of \( A_{\Gamma,\tilde{\beta}} \), where \( \chi \) is defined by the set \( \hat{\alpha}_k^{(i)} \), and \( \tilde{\beta} \) is defined by the set \( \tilde{\alpha}_m^{(i)} \), \( k = 1, \ldots, m_i, i = 1, \ldots, d \), where \( \hat{\alpha}_1^{(i)} = \lambda - \alpha_m^{(i)} \), \( \ldots \), \( \hat{\alpha}_m^{(i)} = \lambda - \alpha_1^{(i)} \), \( i = 1, \ldots, d \). This mapping will be denoted by \( T \).

**Theorem 1.** [KRS02] The mappings \( S \) and \( T \) are functors from the category \( \text{Rep}(A_{\Gamma,\chi}) \) into the categories \( \text{Rep}(A_{\Gamma,\tilde{\beta}}) \) and \( \text{Rep}(A_{\Gamma,\tilde{\beta}'} \) respectively, where \( \tilde{\beta} \), \( \bar{\beta} \) and \( \tilde{\beta} \) are as defined above. Each of these functors being applied twice is the identity functor in \( \text{Rep}(A_{\Gamma,\chi}) \).

7. Given a graph \( \Gamma \), the functors \( S \) and \( T \) define an action on the set of pairs, \((\chi, \lambda)\) as follows: \( S: (\chi, \lambda) \mapsto (\tilde{\chi}, \lambda) \), \( T: (\chi, \lambda) \mapsto (\chi, \tilde{\lambda}) \). Below, we will see that some combinations of these functors leave \( \chi \) unaltered, and in this case we also refer to the action of the corresponding functor on the set \( \Sigma_{\Gamma,\chi} \).

2 **Special characters on extended Dynkin graphs**

We apply the Coxeter functors technique to the study of \(*\)-algebras related to the extended Dynkin graphs. We remark that the functors, \( S \) and \( T \), do not preserve, in general, the character on the graph. However, for the extended Dynkin graphs, there exist special characters described below, which possess some invariance property.

For each graph \( \Gamma \) we write \( \chi_{\Gamma} \) for the special character on \( \Gamma \).

**Proposition 1.** A character \( \chi \) on the extended Dynkin graph \( D_4 \) is invariant with respect to the action of \( TS \) for any \( \lambda \) if and only if \( \chi = \chi_{\hat{D}_4} = (1; 1; 1; 1) \).

A character \( \chi \) on the extended Dynkin graph \( \hat{E}_6 \) is invariant with respect to the action of \((TS)^2 \) for any \( \lambda \) if and only if \( \chi = \chi_{\hat{E}_6} = (1; 2; 1; 2; 1; 2) \).

A character \( \chi \) on the extended Dynkin graph \( \hat{E}_7 \) is invariant with respect to the action of \((TS)^3 \) for any \( \lambda \) if and only if \( \chi = \chi_{\hat{E}_7} = (1; 2; 3; 1; 2; 3; 2) \).

A character \( \chi \) on the extended Dynkin graph \( \hat{E}_8 \) is invariant with respect to the action of \((TS)^5 \) for any \( \lambda \) if and only if \( \chi = \chi_{\hat{E}_8} = (1; 2; 3; 4; 5; 2; 4; 3) \).

Here and below, we consider characters \( \chi \) and \( t\chi, t > 0 \), as identical since they give rise to isomorphic algebras.

**Proof.** It follows by a direct calculation that the characters \( \chi_{\hat{D}_4}, \chi_{\hat{E}_6}, \chi_{\hat{E}_7} \) and \( \chi_{\hat{E}_8} \) are invariant with respect to the corresponding mappings.

For any character \( \chi \) on \( \Gamma \), where \( \Gamma \) is an extended Dynkin graph, introduce \( \omega_\Gamma(\chi) \) as follows:

\[
\omega_{\hat{D}_4}(a; b; c; d) = \frac{1}{2}(a + b + c + d);
\omega_{\hat{E}_6}(a_1, a_2, b_1, b_2, c_1, c_2) = \frac{1}{3}(a_1 + a_2 + b_1 + b_2 + c_1 + c_2);
\omega_{\hat{E}_7}(a_1, a_2, a_3, b_1, b_2, b_2; c) = \frac{1}{4}(a_1 + a_2 + a_3 + b_1 + b_2 + b_3 + 2c);
\omega_{\hat{E}_8}(a_1, a_2, a_3, a_4, a_5, b_1, b_2; c) = \frac{1}{6}(a_1 + a_2 + a_3 + a_4 + a_5 + 2b_1 + 2b_2 + 3c);
\]
Lemma 1. Let $\Gamma$ be an extended Dynkin graph. Then for any $k \in \mathbb{N}$ and for any pair $(\chi, \lambda)$ where $\chi$ is a character and $\lambda \geq 0$ we have

$$(TS)^{\omega_T(\omega_T^{-1})^k}(\chi, \lambda) = (\chi - \omega_T k \gamma\chi_T, \lambda - \omega_T^2 k \gamma),$$

where $\omega_T = \omega_T(\chi_T)$, and $\gamma = \omega_T - \lambda$. Here, $\omega_T$ is equal to 2, 3, 4 and 6 for $\Gamma$ being $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ respectively.

Proof. Following [Ost05], let us norm the character $\chi$ so that $\omega_T(\chi) = \omega_T$, then $\chi$ can be represented as $\chi = \chi_T + \hat{\chi}$, where $\hat{\chi}$ is a (not necessarily positive) character on $\Gamma$, such that $\omega_T(\hat{\chi}) = 0$; also, let us represent $\lambda = \omega_T - \gamma$. Then the statement is a matter of a routine calculations.

To complete the proof of the proposition, it is sufficient to notice that the invariance for all $\lambda$ by the lemma above implies that $\chi$ is proportional to $\chi_T$. \qed

2.1 Structure of the sets $\Sigma_{\Gamma,\chi_\Gamma}$, where $\Gamma$ is an extended Dynkin graph

The results on the structure of $\Sigma_{\Gamma,\chi_\Gamma}$ in the case where $\Gamma$ is an extended Dynkin graph were announced in the earlier paper [MSZ04]. Here we give detailed proofs of these facts.

2.1.1 Algebra related to the graph $\tilde{D}_4$

Theorem 2. The set $\Sigma_{\tilde{D}_4,\chi_{\tilde{D}_4}}$ of those $\lambda$ for which there exist representations of the algebra $A_{\tilde{D}_4,\chi_{\tilde{D}_4}}$ is given by:

$$\Sigma_{\tilde{D}_4,\chi_{\tilde{D}_4}} = \left\{2 \pm \frac{1}{k + s} \mid k = 0, 1, \ldots; s \in \{1/2, 1\}\right\} \cup \{2\}.$$

Proof. Notice first that $\Sigma_{\tilde{D}_4,\chi_{\tilde{D}_4}} \subset [0, 4]$. The functor $S$ maps each point $\lambda$ into $4 - \lambda$, therefore, $\Sigma_{\tilde{D}_4,\chi_{\tilde{D}_4}} \subset [0, 4]$ is symmetric with respect to the point 2. Therefore, there remains to study $\Sigma_{\tilde{D}_4,\chi_{\tilde{D}_4}} \cap [0, 2]$.

First we study one-dimensional representations, i.e., collections of numbers $p_1, p_2, p_3, p_4 \in \{0, 1\}$, such that $p_1 + p_2 + p_3 + p_4 = \lambda \in [0, 2]$. It easily follows that such solutions exist for $\lambda \in \{0, 1, 2\}$.

Since $TS$ is a functor, the set $\Sigma_{\tilde{D}_4,\chi_{\tilde{D}_4}}$ contains also all points obtained as results of the action of the functor $TS$ on $(\chi_{\tilde{D}_4}, 0)$, $(\chi_{\tilde{D}_4}, 1)$, $(\chi_{\tilde{D}_4}, 2)$. For the pair $(\chi_{\tilde{D}_4}, 0)$, we obtain the sequence $\{1 + \frac{2k - 1}{2k + 1}, k = 0, 1, \ldots\}$, and for the pair $(\chi_{\tilde{D}_4}, 1)$ we have the sequence $\{1 + \frac{2k}{2k + 2}, k = 0, 1, \ldots\}$, the pair $(\chi_{\tilde{D}_4}, 2)$ is invariant with respect to $TS$. Therefore, all points listed in the statement of the theorem belong to $\Sigma_{\tilde{D}_4,\chi_{\tilde{D}_4}}$.

It remains to prove that in $\Sigma_{\tilde{D}_4,\chi_{\tilde{D}_4}}$ there are no other points.
Lemma 2. Let $P_1, \ldots, P_n$ be projections such that $\sum_{k=1}^n \alpha_k P_k = \lambda I$ for some positive $\alpha_1, \ldots, \alpha_n$, and some $\lambda \geq 0$. Then $P_j = 0$ for all $j$, for which $\alpha_j > \lambda$.

Proof. Indeed, let $\alpha_j > \lambda$. Multiplying the equality $\sum_{k \neq j} \alpha_k P_k = \lambda I - \alpha_j P_j$ by $P_j$ from the left and right, we get $\sum_{k \neq j} \alpha_k P_k P_j = (\lambda - \alpha_j)P_j$. On the left hand side we have a non-negative operator, while the right-hand side is non-positive, therefore, $P_j = 0$. \hfill $\square$

The lemma implies immediately that there are no points of $\Sigma_{D_4 \times D_4}$ on $(0, 1)$.

Now we show that the interval $(1, \frac{4}{3})$ does not contain points of $\Sigma_{D_4 \times D_4}$. Indeed, since $T$ is a functor, then there would be representations for the pair $(\chi_{D_4}, \frac{\lambda}{\chi})$, but $\frac{\lambda}{\chi} > 4$ for $\lambda \in (1, \frac{4}{3})$. To complete the proof, notice that the images of the interval $[0, \frac{4}{3}]$ (containing only the points $0, 1, 4/3$) under the action of powers of $TS$ cover the whole interval $[0, 2]$. \hfill $\square$

2.1.2 Algebra related to the graph $\tilde{E}_6$

Theorem 3. The set $\Sigma_{\tilde{E}_6 \times \tilde{E}_6}$ of those $\lambda$ for which there exist representations of the algebra $\mathcal{A}_{\tilde{E}_6 \times \tilde{E}_6}, \lambda$ is given by:

$$
\Sigma_{\tilde{E}_6 \times \tilde{E}_6} = \{3 \pm \frac{1}{k + s} \mid k = 0, 1, \ldots; s \in \{1/3, 1/2, 2/3, 1\}\} \cup \{3\}.
$$

Proof. Since for the operators $A_1, A_2, A_3$ we have $\sigma(A_j) \subset \{0, 1, 2\}, \|A_j\| \leq 2$, the estimate $\Sigma_{\tilde{E}_6 \times \tilde{E}_6} \subset [0, 6]$ holds. Also, the functor $S$ establishes the symmetry of $\Sigma_{\tilde{E}_6 \times \tilde{E}_6}$ with respect to the point 3. Therefore, there remains to study $\Sigma_{\tilde{E}_6 \times \tilde{E}_6} \cap [0, 3]$.

For $\lambda \in (0, 1)$ by Lemma 2 there are no representations.

The functor $S$ maps $\lambda \in (0, 1)$ into $\lambda \in (5, 6)$, and $T$ maps $\lambda \in (5, 6)$ into $\lambda \in (1, 3/2)$, which implies that $\Sigma_{\tilde{E}_6 \times \tilde{E}_6} \cap (1, 3/2) = \emptyset$.

The functor $S$ maps $\lambda \in (3/2, 2)$ into $\lambda \in (4, 9/2)$, and $(TS)^2$ maps it into $\lambda \in (-\infty, 0)$, for which there are no solutions. Therefore, $\Sigma_{\tilde{E}_6 \times \tilde{E}_6} \cap (3/2, 2) = \emptyset$.

Finally, the functor $S$ maps $\lambda \in (2, 9/4)$ into $\lambda \in (15/4, 4)$, and $(TS)^2$ maps it into $\lambda \in (6, \infty)$, for which there are no solutions. Therefore, $\Sigma_{\tilde{E}_6 \times \tilde{E}_6} \cap (2, 9/4) = \emptyset$.

Thus we have $(TS)^2$: $(\chi_6, 0) \mapsto (\chi_6, 9/4)$ and $\Sigma_{\tilde{E}_6 \times \tilde{E}_6} \cap [0, 9/4) = \{0, 1, 3/2, 2\}$. We show that for these values of $\lambda$ there exist solutions.

One-dimensional representations are triples of numbers, $a_1, a_2, a_3 \subset \{0, 1, 2\}$ such that $a_1 + a_2 + a_3 = \lambda$. For the segment $[0, 3]$ such representations exist for $\lambda \in (0, 1, 2, 3)$.

For $\lambda = 3/2$ there exists a representation as well. Indeed, we need to show that there exist projections $P_1, P_2, Q_1, Q_2, R_1, R_2$, such that

$$
P_1 + 2P_2 + Q_1 + 2Q_2 + R_1 + 2R_2 = \frac{3}{2}I.
$$

By Lemma 2 we have $P_2 = Q_2 = R_2 = 0$, and the problem reduces to $P_1 + Q_1 + R_1 = \frac{3}{2}I$. The corresponding character has the form $\chi = (1; 1; 1)$, and the functor $T$ maps the pair $7$
(χ, 3) into (χ, 3). The corresponding problem is $P'_1 + Q'_1 + R'_1 = 3I$, which has one-dimensional irreducible representations $P'_1 = Q'_1 = R'_1 = 1$. The corresponding projections are

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_1 = \frac{1}{4} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}, \quad R_1 = \frac{1}{4} \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix},$$

for which $P_1 + Q_1 + R_1 = \frac{3}{2}I$.

Since the images of $[0, 9/4)$ under the action of powers of $(TS)^3$ cover the whole interval $[0, 3)$, we conclude that $\Sigma_{E_7, X_{E_7}} \cap [0, 3)$ consists of the iterations of the set $\{0, 1, 3/2, 2\}$ under the mapping $\lambda \mapsto 2 + \frac{1}{4-\lambda}$.

\[ \square \]

### 2.1.3 Algebra related to the graph $\tilde{E}_7$

**Theorem 4.** The set $\Sigma_{\tilde{E}_7, X_{\tilde{E}_7}}$ of those $\lambda$ for which there exist representations of the algebra $A_{\tilde{E}_7, X_{\tilde{E}_7}, \lambda}$ is given by:

$$\Sigma_{\tilde{E}_7, X_{\tilde{E}_7}} = \left\{ 4 \pm \frac{1}{k} + s \mid k = 0, 1, \ldots; s \in \{1/4, 1/3, 1/2, 2/3, 3/4, 1\} \right\} \cup \{4\}.$$

**Proof.** Similarly to the proof of the previous theorem, we have that $\Sigma_{\tilde{E}_7, X_{\tilde{E}_7}} \subset [0, 8]$ and it is symmetric around 4. Consider the structure of $\Sigma_{\tilde{E}_7, X_{\tilde{E}_7}} \cap [0, 4]$.

For $\lambda = 4$ there exist one-dimensional representations, and the interval $[0, 4)$ is covered by the images of $[0, 16/5)$ under the mapping $\lambda \mapsto 2 + \frac{1}{4-\lambda}$, which arises from the action of $(TS)^3$. Therefore, there remains to study the structure of $\Sigma_{\tilde{E}_7, X_{\tilde{E}_7}} \cap [0, 16/5)$.

Like above, $\Sigma_{\tilde{E}_7, X_{\tilde{E}_7}} \cap (0, 1) = \emptyset$. For the interval $(1, 2)$ we have the following argument. Since $\lambda < 2$, for projections $P_1, P_2, P_3, Q_1, Q_2, Q_3, R_1$ such that

$$P_1 + 2P_2 + 3P_3 + Q_1 + 2Q_2 + 3Q_3 + 2R_1 = \lambda I$$

we have by Lemma 2 that $P_2 = P_3 = Q_2 = Q_3 = R_1 = 0$, and the solutions are described by $P_1 + Q_1 = \lambda I$. But this is possible for $\lambda \in \{0, 1, 2\}$ only. Therefore $\Sigma_{\tilde{E}_7, X_{\tilde{E}_7}} \cap (1, 2) = \emptyset$.

Applying $S$ and $(TS)^3$ to $\lambda \in (0, 1) \cup (1, 2)$ we conclude that $\Sigma_{\tilde{E}_7, X_{\tilde{E}_7}} \cap (5/2, 8/3) = \emptyset$ and $\Sigma_{\tilde{E}_7, X_{\tilde{E}_7}} \cap (2, 5/2) = \emptyset$.

The functor $S$ maps $\lambda \in (8/3, 3)$ into $\lambda \in (5, 16/3)$, and $(TS)^3$ maps it into $\lambda \in (-\infty, 0)$, for which there are no solutions. Therefore $\Sigma_{\tilde{E}_7, X_{\tilde{E}_7}} \cap (8/3, 3) = \emptyset$.

The functor $S$ maps $\lambda \in (3, 16/5)$ into $\lambda \in (24/5, 5)$, and $(TS)^3$ maps it into $\lambda \in (8, \infty)$, for which there are no solutions. Therefore, we have $\Sigma_{\tilde{E}_7, X_{\tilde{E}_7}} \cap (3, 16/5) = \emptyset$.

Thus we see that $\Sigma_{\tilde{E}_7, X_{\tilde{E}_7}} \cap [0, 16/5) = \{0, 1, 2, \frac{5}{2}, \frac{8}{3}, 3\}$.

For the points 0, 1, 2, 3 there exist one-dimensional representations which can easily be calculated.

For $\lambda = \frac{5}{2}$ and $\lambda = \frac{8}{3}$, since $\lambda < 3$, we have $P_3 = Q_3 = 0$ in (1), and therefore the equation reduces to $P_1 + 2P_2 + Q_1 + 2Q_2 + 2R_1 = \lambda I$. For $\lambda = 5/2$, applying the
functor $S(TS)^3$ to the pair $(\chi, 5/2)$, $\chi = (1, 2; 1, 2; 2)$, we get the pair $(\chi, 1)$, which, since $1 < 2$, leads to the equation $P_1' + Q_1' = I$, which has two one-dimensional irreducible representations.

For $\lambda = 8/3$, the same functor maps the pair $(\chi, 8/3)$ into the pair $(\chi, 0)$, and the corresponding equation has a single trivial solution.

Therefore, the whole $\Sigma_{(TS)^3} \cap (0, 4)$ can be obtained as images of the set $\{0, 1, 2, \frac{5}{2}, \frac{8}{3}, 3\}$ under the mapping $\lambda \mapsto 3 + \frac{1}{s\lambda}$, arising from the action of the functor $(TS)^3$.  

\section{2.1.4 Algebra related to the graph $\tilde{E}_8$}

\textbf{Theorem 5.} The set $\Sigma_{\tilde{E}_8, X_{\tilde{E}_8}}$ of those $\lambda$ for which there exist representations of the algebra $A_{\tilde{E}_8, X_{\tilde{E}_8}, \lambda}$ is given by:

$$\Sigma_{\tilde{E}_8, X_{\tilde{E}_8}} = \left\{ 6 \pm \frac{1}{k + s} \mid k = 0, 1, \ldots; \right.$$  

$$s \in \{1/6, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 5/6, 1\} \cup \{6\}.$$

\textbf{Proof.} As before, we have $\Sigma_{\tilde{E}_8, X_{\tilde{E}_8}} \subset [0, 12]$ and $\Sigma_{\tilde{E}_8, X_{\tilde{E}_8}}$ is symmetric around 6. Also, in the point $\lambda = 6$ there are representations (e.g., one-dimensional). Again, the structure of $\Sigma_{\tilde{E}_8, X_{\tilde{E}_8}} \cap [0, 6)$ is determined by the structure of $\Sigma_{\tilde{E}_8, X_{\tilde{E}_8}} \cap [0, 5 + 1/7)$, since $(TS)^5$ maps $(\chi_8, 0)$ into $(\chi_8, 5 + 1/7)$.

Consider the families of projections $P_1, \ldots, P_5, Q_1, Q_2, R$, such that $P_1 + 2P_2 + 3P_3 + 4P_4 + 5P_5 + 2Q_1 + 4Q_2 + 3Q_3 = \lambda I$ and $P_jP_k = 0, j \neq k, Q_1Q_2 = 0$.

According to Lemma 2, for $\lambda < 3$ we have $P_3 = P_4 = P_5 = Q_2 = R = 0$, or $P_1 + 2P_2 + 2Q_1 = \lambda I$, which implies that $Q_1$ commutes with $P_1, P_2$; all irreducible representations of such a family are one-dimensional and exist for $\lambda \in \{0, 1, 2, 3, 4\}$ only.

For the case $3 < \lambda < 4$, we have $P_4 = P_5 = Q_2 = 0$, or $P_1 + 2P_2 + 3P_3 + 2Q_1 + 3R = \lambda I$. Therefore, $\Sigma_{\tilde{E}_8, X_{\tilde{E}_8}} \cap (3, 4) = \Sigma_{D_{6, \lambda}} \cap (3, 4)$, where $\chi = (1, 2, 3; 2; 3)$.

\textbf{Proposition 2.} $\Sigma_{D_{6, (1, 2, 3; 2; 3)}} \cap (3, 4) = \{7/2\}$.

\textbf{Proof.} Acting by $STS$ on $((1, 2, 3; 2; 3), \lambda)$ we get $((1, 2, 7 - \lambda; 6 - \lambda; 5 - \lambda), 10 - 2\lambda)$. For $\lambda \in (3, 4)$ we have $7 - \lambda > 10 - 2\lambda$, therefore, $\Sigma_{D_{6, (1, 2, 3; 2; 3)}} \cap (3, 4)$ consists of those $\lambda$ for which there exist solutions for $P_1 + 2P_2 + (6 - \lambda)Q_1 + (5 - \lambda)R = (10 - 2\lambda)I$. Applying $STS$ to this set of projections we get the pair $((1, 2; \lambda - 3; \lambda - 2), 2\lambda - 6)$. For $\lambda \in (3, 4)$ we have $\lambda - 2 > 2\lambda - 6$, and the relation $P_1 + 2P_2 + (\lambda - 3)Q_1 = (2\lambda - 6)I$, which implies that the projections commute and that the only value of $\lambda$, for which there exists a solution is $\lambda = 7/2$.

Thus, $\Sigma_{\tilde{E}_8, X_{\tilde{E}_8}} \cap (3, 4) = \{7/2\}$. Further, $(TS)^5S$ maps the interval $(3, 4)$ into the interval $(4, 9/2)$, and $\lambda = 7/2$ into $5 - 2/3 = 13/3$. The interval $(0, 3)$ is mapped by the same functor into $(9/2, 24/5) = (5 - 1/2, 5 - 1/5)$, and the points $\{0, 1, 2, 3\}$ are mapped
into the points \( \{5 - 1/5, 5 - 1/4, 5 - 1/3, 5 - 1/2\} \) respectively. Similarly to the previous theorem we get \( \Sigma_{\tilde{E}_8,\tilde{E}_8} \cap (24/5,5) = \emptyset \) and \( \Sigma_{\tilde{E}_8,\tilde{E}_8} \cap (5,36/7) = \emptyset \).

Therefore, \( \Sigma_{\tilde{E}_8,\tilde{E}_8} \cap [0,36/7) = \{0,1,2,3,7/2,4,13/3,9/2,14/3,24/5,5\} \), and we get the result by iterating the mapping \( \lambda \mapsto 5 + \frac{1}{7-\lambda} \) on this set, adding \( \lambda = 6 \) and applying the symmetry around \( \lambda = 6 \).

### 2.2 \(*\)-Representations of the algebras \( \mathcal{A}_\Gamma,\chi_\Gamma \) where \( \Gamma \) is an extended Dynkin graph

For \( \mathcal{A}_{\tilde{D}_4,\tilde{X}_{\tilde{D}_4}} \) we give explicit formulas for discrete series (singular) representations as well as for fixed point (regular) representations.

For other algebras the Coxeter functions give an algorithm for construction of representations of discrete (singular) series including their generalized dimensions.

For the fixed point (regular) case we give explicit formulas (except for the case of the algebra \( \mathcal{A}_{\tilde{E}_8,\tilde{E}_8} \)).

#### 2.2.1 Representations of the algebra related to the graph \( \tilde{D}_4 \)

Representations of the algebra \( \mathcal{A}_{\tilde{D}_4,\tilde{X}_{\tilde{D}_4}} \) are generated by quadruples of projections, \( P_1, P_2, P_3, P_4 \) such that \( P_1 + P_2 + P_3 + P_4 = \lambda I, \lambda \geq 0 \).

There are five one-dimensional representations: one representation corresponding to \( \lambda = 0 \) \((P_1 = P_2 = P_3 = P_4 = 0)\) and four representations corresponding to \( \lambda = 1 \) \((P_1 = 1, P_2 = P_3 = P_4 = 0 \text{ and their permutations})\).

Applying the functors, we get five infinite series of irreducible representations.

To write the explicit formulas for the representations of these series, we introduce the following projections in \( \mathbb{C}^2 \): \( Q_{l,m} \) and \( R_{l,m} \).

\[
Q_{l,m} = \frac{1}{m} \left( \begin{array}{c}
\frac{l}{\sqrt{l(l-m)}} \\
\frac{\sqrt{l(l-m)}}{m-l}
\end{array} \right),
\]

\[
R_{l,m} = \frac{1}{m} \left( \begin{array}{c}
\frac{l}{\sqrt{l(l-m)}} \\
\frac{-\sqrt{l(l-m)}}{m-l}
\end{array} \right), \quad 0 \leq l \leq m.
\]

Then the representations can be given as follows:

0-series: \( \dim H = n = 2k + 1, k \geq 0, \lambda = 2 \pm \frac{2}{n} \).

The projections are:

\[
P_1 = Q_{n-1,n} \oplus Q_{n-3,n} \oplus \cdots \oplus Q_{2,n} \oplus 0,
\]

\[
P_2 = R_{n-1,n} \oplus R_{n-3,n} \oplus \cdots \oplus R_{2,n} \oplus 0,
\]

\[
P_3 = 0 \oplus Q_{n-2,n} \oplus Q_{n-4,n} \oplus \cdots \oplus Q_{1,n},
\]

\[
P_4 = 0 \oplus R_{n-2,n} \oplus R_{n-4,n} \oplus \cdots \oplus R_{1,n},
\]

\( H = \bigoplus_{i=1}^{k} \mathbb{C}^2 \oplus \bigoplus_{i=1}^{k} \mathbb{C}^1 \).

For \( \lambda = 1 \):

\[
P_1 = Q_{n-1,n} \oplus Q_{n-3,n} \oplus \cdots \oplus Q_{2,n} \oplus 0,
\]

\[
P_2 = R_{n-1,n} \oplus R_{n-3,n} \oplus \cdots \oplus R_{2,n} \oplus 0,
\]

\[
P_3 = 0 \oplus Q_{n-2,n} \oplus Q_{n-4,n} \oplus \cdots \oplus Q_{1,n},
\]

\[
P_4 = 0 \oplus R_{n-2,n} \oplus R_{n-4,n} \oplus \cdots \oplus R_{1,n},
\]

\( H = \bigoplus_{i=1}^{k} \mathbb{C}^2 \oplus \bigoplus_{i=1}^{k} \mathbb{C}^2 \).

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for which \( P_1 + P_2 + P_3 + P_4 = 2 - \frac{2}{n} \), and the images of these projections under the action of the \( S \) functors, \( P_j' = I - P_j, j = 1, \ldots, 4 \), for which \( P'_1 + P'_2 + P'_3 + P'_4 = 2 + \frac{2}{n} \).

1-series: \( \dim H = n \geq 0 \), \( \lambda = 2 \pm \frac{1}{n} \) (thus a total of 4 series). For \( n = 2k \)

\[
P_1 = 0 \oplus Q_{2k-2,4k} \oplus Q_{2k-4,4k} \oplus \cdots \oplus Q_{2,4k} \oplus 0, \\
P_2 = 4k \oplus R_{2k-2,4k} \oplus R_{2k-4,4k} \oplus \cdots \oplus R_{2,4k} \oplus 0, \\
H = \mathbb{C}^1 \oplus \bigoplus_{k=1}^{n} \mathbb{C}^2, \\
P_3 = Q_{2k-1,4k} \oplus Q_{2k-3,4k} \oplus \cdots \oplus Q_{1,4k}, \\
P_4 = R_{2k-1,4k} \oplus R_{2k-3,4k} \oplus \cdots \oplus R_{1,4k}, \\
H = \bigoplus_{k=1}^{n} \mathbb{C}^2.
\]

For \( n = 2k + 1 \)

\[
P_1 = 4k + 2 \oplus Q_{2k-1,4k+2} \oplus Q_{2k-3,4k+2} \oplus \cdots \oplus Q_{1,4k+2}, \\
P_2 = 0 \oplus R_{2k-1,4k+2} \oplus R_{2k-3,4k+2} \oplus \cdots \oplus R_{1,4k+2}, \\
H = \mathbb{C}^1 \oplus \bigoplus_{k=1}^{n} \mathbb{C}^2, \\
P_3 = Q_{2k,4k+2} \oplus Q_{2k-2,4k+2} \oplus \cdots \oplus Q_{2,4k+2} \oplus 0, \\
P_4 = R_{2k,4k+2} \oplus R_{2k-2,4k+2} \oplus \cdots \oplus R_{2,4k+2} \oplus 0, \\
H = \bigoplus_{k=1}^{n} \mathbb{C}^2. 
\]

Other 1-series are obtained by cyclic permutation of the projections above.

For the projections constructed above we have \( P_1 + P_2 + P_3 + P_4 = 2 - \frac{1}{n} \). Applying the \( S \) functor, we append to the list the projections \( P_j' = I - P_j, j = 1, \ldots, 4 \), for which \( P'_1 + P'_2 + P'_3 + P'_4 = 2 + \frac{1}{n} \).

Irreducible quadruples of projections, such that \( P_1 + P_2 + P_3 + P_4 = 2I \) are described in [OS99]. These representations are given by the following proposition.

**Proposition 3.** Any irreducible four-tuple of projections, such that \( P_1 + P_2 + P_3 + P_4 = 2I \) is unitary equivalent to one of the following:

1. six one-dimensional representations, \( P_j = p_j \in \{0, 1\}, j = 1, \ldots, 4 \), where two of the \( p_j \) are zero;
2. family of two-dimensional representations of the form

\[
P_1 = \frac{1}{2} \begin{pmatrix} 1 + a & -b - ic \\ -b + ic & 1 - a \end{pmatrix}, \\
P_2 = \frac{1}{2} \begin{pmatrix} 1 - a & b + ic \\ b + ic & 1 + a \end{pmatrix}, \\
P_3 = \frac{1}{2} \begin{pmatrix} 1 - a & -b + ic \\ -b - ic & 1 + a \end{pmatrix}, \\
P_4 = \frac{1}{2} \begin{pmatrix} 1 + a & b + ic \\ b - ic & 1 - a \end{pmatrix},
\]

where \( a^2 + b^2 + c^2 = 1 \), and either \( a > 0, b > 0, c \in (-1, 1) \), or \( a = 0, b > 0, c > 0 \), or \( a > 0, b = 0, c > 0 \).

**Proof.** Introduce the following operators: \( C_1 = I - P_2 - P_3, C_2 = I - P_1 - P_3, C_3 = I - P_1 - P_2 \). Then the relation \( P_1 + P_2 + P_3 + P_4 = 2I \) is equivalent to the following relations for \( C_1, C_2, C_3 \): \( \{C_1, C_2\} = \{C_1, C_3\} = \{C_2, C_3\} = 0, C_1^2 + C_2^2 + C_3^2 = I \). Then the statement follows from the description of triples of self-adjoint operators which pairwise anticommute. \( \square \)
2.2.2 Representations of the algebra related to the graph $\tilde{E}_6$

For $\lambda \neq 3$, all irreducible representations of $\mathcal{A}_{\tilde{E}_6 \times \tilde{E}_6, \lambda}$ can be obtained from the simplest ones (one- or two-dimensional) using the Coxeter functors described above. However, the procedure involves lots of routine calculations and the result does not have such a nice form as for the algebra related to $D_4$.

Irreducible representations of $\mathcal{A}_{\tilde{E}_6 \times \tilde{E}_6, 3}$ are described in [Mel03]. The representations are determined by triples of self-adjoint operators whose spectra lie in $\{0, 1, 2\}$ and their sum is equal to $3I$. Shifting these operators by a scalar, we get the following problem: describe he irreducible triples, $A_1, A_2, A_3$ for which

$$A_i^3 = A_i, \quad i = 1, 2, 3; \quad A_1 + A_2 + A_3 = 0.$$

**Theorem 6.** Any irreducible triple of self-adjoint operators, $A_1, A_2, A_3$ in a separable Hilbert space for which $A_1 + A_2 + A_3 = 0$ and $\sigma(A_i) \subset \{-1, 0, 1\}, i = 1, 2, 3$, is unitary equivalent to one of the following non-equivalent irreducible triple of operators:

1. Seven one-dimensional representations:
   
   (a) $A_1 = A_2 = A_3 = 0$,
   
   (b) $A_1 = -1, A_2 = 1, A_3 = 0$, and any of their permutations.

2. One two-dimensional representation:

   $$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, \quad A_3 = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$$

3. A family of three-dimensional representations:

   $$A_1 = \begin{pmatrix} 0 & \lambda_1 & i\lambda_3 \\ \lambda_1 & 0 & \lambda_2 \\ -i\lambda_3 & \lambda_2 & 0 \end{pmatrix},$$

   $$A_2 = -\frac{1}{2} \begin{pmatrix} 0 & \lambda_1(1 + i\sqrt{3}) & \lambda_3(i + \sqrt{3}) \\ \lambda_1(1 - i\sqrt{3}) & 0 & \lambda_2(1 + i\sqrt{3}) \\ \lambda_3(-i + \sqrt{3}) & \lambda_2(1 - i\sqrt{3}) & 0 \end{pmatrix},$$

   $$A_3 = \frac{1}{2} \begin{pmatrix} 0 & \lambda_1(-1 + i\sqrt{3}) & \lambda_3(-i + \sqrt{3}) \\ \lambda_1(-1 - i\sqrt{3}) & 0 & \lambda_2(-1 + i\sqrt{3}) \\ \lambda_3(i + \sqrt{3}) & \lambda_2(-1 - i\sqrt{3}) & 0 \end{pmatrix},$$

where $\lambda_i \in \mathbb{R}, i = 1, 2, 3; \lambda_1, \lambda_2 > 0, \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$, and either $|\lambda_3| \neq \lambda_1 = \lambda_2$, or $|\lambda_3| < \lambda_1, |\lambda_3| < \lambda_2$ and $\lambda_1 \neq \lambda_2$.

**Proof.** We give a sketch of the proof. Let us take an element $x$ in the algebra such that $A_k = e_k x + e_k x^*$, $k = 1, 2, 3$, where $\epsilon = (-1 + i\sqrt{3})/2$. Such element is unique, and together with $x^*$ they generate the whole algebra $\mathcal{A}_{\tilde{E}_6 \times \tilde{E}_6, 3}$. One can show that this element is centered, and apply the results of [Ost96] and the fact that $x^3$ belongs to the center of the algebra (see [Mel03] for the details).
2.2.3 Representations of the algebra related to the graph \( \tilde{E}_7 \)

As above, we restrict ourselves to the case \( \mathcal{A}_{\tilde{E}_7, x_{\tilde{E}_7}, A} \). In the case of \( \lambda \neq 4 \) all irreducible representations are obtained as a result of the action of the Coxeter functors on degenerate representations (which are in fact representations of a finite dimensional subalgebra).

The irreducible representations of \( \mathcal{A}_{\tilde{E}_7, x_{\tilde{E}_7}, A} \) are described in [Ost04]. They are determined by triples of self-adjoint operators \( A_1, A_2, A_3 \) such that the spectrum of \( A_1 \) and \( A_2 \) belongs to \( \{0, 1, 2, 3\} \), and the spectrum of \( A_3 \) belongs to \( \{0, 2\} \) and \( A_1 + A_2 + A_3 = 4I \), or equivalently, pairs of self-adjoint operators \( A = A_1 - 3/2I, B = A_2 - 3/2I \) having their spectrum in the set \( \{\pm 1/2, \pm 3/2\} \) such that \( (A + B)^2 = I \).

**Theorem 7.** The following pairs are representations of the algebra \( \mathcal{A}_{\tilde{E}_7, x_{\tilde{E}_7}, A} \)

\[
A = \frac{1}{8\phi} \begin{pmatrix}
-16 - \beta & 2\lambda & \sqrt{4\phi^4 - \beta^2} & 0 \\
2\lambda & 16 + \beta & 0 & -\sqrt{4\phi^4 - \beta^2} \\
\sqrt{4\phi^4 - \beta^2} & 0 & -16 + \beta & 2\lambda \\
0 & -\sqrt{4\phi^4 - \beta^2} & 2\lambda & 16 - \beta
\end{pmatrix}
\]

\[
B = \frac{1}{8\phi} \begin{pmatrix}
16 - \beta & 2\lambda\omega & -\sqrt{4\phi^4 - \beta^2} & 0 \\
2\lambda\bar{\omega} & -16 + \beta & 0 & \sqrt{4\phi^4 - \beta^2} \\
-\sqrt{4\phi^4 - \beta^2} & 0 & 16 + \beta & 2\lambda\omega \\
0 & \sqrt{4\phi^4 - \beta^2} & 2\lambda\bar{\omega} & -16 - \beta
\end{pmatrix}
\]

where \( \lambda^2 = -\phi^4 + 20\phi^2 - 64, \beta^2 = 16\phi^2 - \lambda^2(1 + \omega)(1 + \bar{\omega}), |\omega| = 1, \phi \in [2, 4] \).

For \( \phi = 2 \) the representation decomposes into the direct sum of four one-dimensional representations, \( A = \pm 1/2, B = \mp 3/2 \) and \( A = \pm 3/2, B = \mp 1/2 \).

For \( \phi = 2\sqrt{2}, \omega = 1 \) the representation decomposes into the direct sum of two one-dimensional representations \( A = B = \pm 1/2 \) and one irreducible two-dimensional representation

\[
A = \begin{pmatrix}
-\sqrt{2} & 1/2 \\
1/2 & \sqrt{2}
\end{pmatrix}, \quad B = \begin{pmatrix}
\sqrt{2} & 1/2 \\
1/2 & -\sqrt{2}
\end{pmatrix}
\]

with the spectrum of \( A \) and \( B \) equal to \( \{-3/2, 3/2\} \).

For \( \phi = 4 \) the representation decomposes into the direct sum of two irreducible two-dimensional representations,

\[
A = \begin{pmatrix}
1/2 & 0 \\
0 & -3/2
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & \sqrt{3}/2 \\
\sqrt{3}/2 & 1
\end{pmatrix}
\]

\[\sigma(A) = \{1/2, -3/2\}, \quad \sigma(B) = \{-1/2, 3/2\}\]

and

\[
A = \begin{pmatrix}
-1/2 & 0 \\
0 & 3/2
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & -\sqrt{3}/2 \\
-\sqrt{3}/2 & -1
\end{pmatrix}
\]

\[\sigma(A) = \{-1/2, 3/2\}, \quad \sigma(B) = \{1/2, -3/2\}\]
For all other pairs \((\phi, \omega)\), \(\phi \in (2, 4)\), \(|\omega| = 1\) the representation is irreducible with \(\sigma(A) = \sigma(B) = \{\pm 1/2, \pm 3/2\}\).

Listed above are all irreducible representations up to a unitary equivalence.

**Proof.** Again, we give only a sketch of the proof. First, using the relations in the algebra, we show that the operators \(A\) and \(B\) can be represented in the following form:

\[
A^2 = \frac{1}{4} \begin{pmatrix}
(5 + 2\lambda)I & -\mu I \\
-\mu I & (5 - 2\lambda)I
\end{pmatrix}, \quad B^2 = \frac{1}{4} \begin{pmatrix}
(5 - 2\lambda)I & -\mu I \\
-\mu I & (5 + 2\lambda)I
\end{pmatrix}.
\]

Here \(\mu, \lambda \geq 0\) and \(\mu^2 + \lambda^2 = 16\). Using this decomposition, we represent \(A\) and \(B\) in the following form

\[
A = \frac{1}{2} U_1 \begin{pmatrix}
A_1 & 0 \\
0 & 3A_2
\end{pmatrix} U_1^*, \quad B = \frac{1}{2} U_2 \begin{pmatrix}
B_1 & 0 \\
0 & 3B_2
\end{pmatrix} U_2^*,
\]

where \(U_1\) and \(U_2\) are unitary matrices which are written in the explicit form. Then, taking into account the relations between the operators \(A, B\) and conditions on their spectrum, after some calculations we obtain the needed formulas (see [Ost04] for the details).

### 2.2.4 Representations of the algebra related to the graph \(\tilde{E}_8\)

As above, all representations of \(A_{\tilde{E}_8, \chi_{\tilde{E}_8, \lambda}}\) for \(\lambda \neq 6\) (discrete series) can be obtained from the simplest ones applying the Coxeter functors.

The formulas for irreducible representations of \(A_{\tilde{E}_8, \chi_{\tilde{E}_8, \lambda}}\) are unknown. However, as shown in [Mel05], these representations are at most six-dimensional, as in the above cases they are parametrized by points of a two-dimensional sphere so that in all points but three, the representation has maximal dimension, and in the three points it decomposes into irreducible representations of smaller dimensions.

### 3 General characters on extended Dynkin graphs

#### 3.1 The structure of the set \(\Sigma_{\Gamma, \chi}\)

The following statement [Yus05] describes the structure of the set \(\Sigma_{\Gamma, \chi}\) for general character \(\chi\) on \(\Gamma\).

**Theorem 8.** Let \(\Gamma\) be an extended Dynkin graph. Then the following holds.

1. The set \(\Sigma_{\Gamma, \chi}\) is finite or countable. The set \(\Sigma_{\Gamma, \chi}\) is infinite if and only if all components \(\chi_k < \omega_T(\chi)\) or \(\hat{\chi}_k < \omega_T(\hat{\chi})\), where \(\hat{\chi}\) is given by \(T\).

2. If \(\Sigma_{\Gamma, \chi}\) is infinite, it contains a unique limit point \(\omega_T(\chi)\).
3.2 Representations

Let \( \Gamma \) be an extended Dynkin graph, and let \( \chi \) be a character on it.

**Theorem 9.** All irreducible families of operators corresponding to extended Dynkin graphs are finite-dimensional.

**Proof.** Let \( \pi \) be an irreducible representation of the algebra \( A_{\Gamma, \chi, \lambda} \), where \( \Gamma \) is an extended Dynkin graph. We consider two cases.

1. Let \( \lambda = \omega_{\Gamma}(\chi) \). It is shown in [Mel05, MSV05] that the corresponding algebra is finite-dimensional over its center, and therefore, it is an algebra with polynomial identity (PI-algebra). This implies that the dimensions of all its irreducible representations are bounded.

2. Let \( \lambda < \omega_{\Gamma}(\chi) \). We proceed as follows. We apply the \((ST)^n\) functors to the representation of the algebra corresponding to the pair \((\chi, \lambda)\) to get representations of the algebras corresponding to other pairs \((\chi_n, \lambda_n)\) and show that at some step either there cannot exist representations (in this case, there are no representations for \((\chi, \lambda)\)), or the representation is an obvious extension of the one of a subgraph (such a subgraph is a Dynkin graph, and the corresponding algebra is finite dimensional, therefore it has a finite number of representations all of which are finite-dimensional). Then the initial representation \( \pi \) is obtained from some finite-dimensional representation of \( A_{\Gamma, \chi_n, \lambda_n} \) as a result of applying the \((TS)^n\) functor, and therefore, is finite-dimensional as well.

Notice first that if some of the coefficients of \( \chi \) are larger than \( \lambda \), then the corresponding projections are zero (Lemma 2).

In the case where some of the coefficients of \( \chi \) are equal to \( \lambda \), the corresponding projection commutes with all other projections and therefore, is either the identity (in this case all other projections are zero), or zero.

Thus, in both these cases, the representation is in fact a representation of the subalgebra in \( A_{\Gamma, \chi, \lambda} \) corresponding to some subgraph.

Now let all coefficients of \( \chi_k, k \leq n \) be positive, and some coefficient of \( \chi_{n+1} \) be negative or zero. Taking into account the way the functors act on characters, we easily see that this means that the corresponding coefficient of \( \chi_n \) is larger or equal \( \lambda_n \).

To complete the proof, it is now sufficient to show that for any \( \lambda < \omega_{\Gamma}(\chi) \) there exists such a number \( n \) that some coefficient of \( \chi_n \) is negative or zero. But this immediately follows from Lemma 1, which completes the proof in the case \( \lambda < \omega_{\Gamma}(\chi) \).

3. Let \( \lambda > \omega_{\Gamma}(\chi) \). By applying the \( S \) functor to the pair \((\chi, \lambda)\), we get a pair \((\chi', \lambda')\) with \( \lambda' < \omega_{\Gamma}(\chi') \) and the arguments above apply.

The following properties of representations can also be easily proven:

— the dimensions of the representations grow to infinity as \( \lambda \to \omega \); this follows from the formulas for the \( T \) functor;

— there exists only a finite number of representations at each point \( \lambda \) of discrete series; indeed, all of them are obtained from representations of a smaller finite dimensional subalgebra;
— for regular points (λ = ωΓ(χ)) the dimensions are bounded, because the corresponding algebra is a PI-algebra.

Recall that the rigidity index [Kat96, SV99] of a family of operators \( A_j, j = 1, \ldots, k \) in an \( n \)-dimensional space is

\[
r = n^2(2 - k) + \sum_{j=1}^{k} c(A_j)
\]

where \( c(A_j) \) is the dimension of the centralizer of \( A_j \).

**Corollary 1.** The rigidity index is equal to 2 for all irreducible representations of \( A_{Γ,χ,λ} \) where \( Γ \) is an extended Dynkin graph and \( λ \neq ω(χ) \).

**Proof.** Indeed, the rigidity index is preserved by \( T \) and \( S \) functors. From the proof of Theorem 9 it follows that any irreducible representation can be obtained from one-dimensional ones under the action of the Coxeter functors (in fact, it was shown that any irreducible representation is obtained from the representation of some algebra related to an ordinary Dynkin graph, but this process can be iterated for the subalgebra until a one-dimensional space is reached). Now the result follows from the directly verified fact that \( r = 2 \) for one-dimensional representations.

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